Standard Method of Differentiation

$$\frac{d}{dx}\mathrm{x}^{\mathrm{n}} = \mathrm{nx}^{\mathrm{n}-1}$$

The symbol $\frac{d}{dx}$ here means that we are differentiating (with respect to x, as the equation will be made up of x-terms). This rule is extremely useful for differentiating expressions and is much faster than differentiating from first principles. Differentiation is the process of finding the derivative, or finding an expression for the gradient of a function.

For example, let

$$f(x) = 5x^3 + 4x^2 + 3x + 2$$

We can apply this rule to find $f'(x)$

$$f'(x) = 3 \times 5x^{3-1} + 2 \times 4x^{2-1} + 1 \times 3x^{1-1} + 0$$

Simplifying we have

$$f'(x) = 15x^2 + 8x^1 + 3x^0$$

Or

$$f'(x) = 15x^2 + 8x + 3$$

This is because $x^0 = 1$ and $x^1 = x$.

If we input $x = 5$ into the expression $f'(x)$ then we find $f'(5) = 15 \times 5^2 + 8 \times 5 + 3 = 418$, so the gradient of the curve $y = f(x)$ is 418 when $x = 5$.

Proof

This method of proof is known as proof by induction. This is a proof in which we establish that if $n = k$ is true, then $n = k + 1$ is true

Suppose that the above equation is true when $n = k$

$$\frac{d}{dx}\mathrm{x}^{\mathrm{k}} = \mathrm{kx}^{\mathrm{k}-1}$$

Then if $n = k + 1$

$$\frac{d}{dx}\mathrm{x}^{\mathrm{k}+1} = \frac{d}{dx}(\mathrm{x}^{\mathrm{k}} \times x)$$

We can then use the product rule[1]

$$\frac{d}{dx}(\mathrm{x}^{\mathrm{k}} \times x) = x \times \frac{d}{dx}x^k + x^k \times \frac{d}{dx}x$$

We already know the derivative of x^k and the derivative of x can be proven from first principles

$$\frac{d}{dx}(\mathrm{x}^{\mathrm{k}} \times x) = x \times \mathrm{kx}^{\mathrm{k}-1} + x^k$$

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[1] The product rule is used to differentiate an expression where terms are multiplied together. It says that $\frac{d}{dx}uv = u\frac{dv}{dx} + v\frac{du}{dx}$, so if we let $u = x^k$ and $v = x$ then $\frac{d}{dx}(\mathrm{x}^{\mathrm{k}} \times x) = x \times \frac{d}{dx}x^k + x^k \times \frac{d}{dx}x$

This can be simplified

$$\frac{d}{dx}\mathrm{x}^{\mathrm{k+1}} = kx^k + x^k$$

Factorising out x^k

$$\frac{d}{dx}\mathrm{x}^{\mathrm{k+1}} = (k+1)x^k$$

So if the result is true for k, it is true for k+1
If n = 1

$$\frac{d}{dx}x = 1 = 1 \times x^0$$

So this is equation is true for all positive integers.

But what about instances where n is not a positive integer? For example, how would one differentiate $\frac{1}{x}$? ($\frac{1}{x} = x^{-1}$)

There are three instances we need to consider. When $n = 0$, when n is negative ($n = -m$) or when n is a fraction $n = \frac{p}{q}$.

When $n = 0$

$$\frac{d}{dx}x^0 = \frac{d}{dx}1 = 0 \times x^{0-1} = 0$$

This result agrees with what we find by differentiating from first principles

When $n = -m$

$$\frac{d}{dx}x^{-m} = \frac{d}{dx}(x^{-1})^m$$

Using the chain rule[2]

$$\frac{d}{dx}(x^{-1})^m = m(x^{-1})^{m-1} \times -x^{-2}$$

$$\frac{d}{dx}(x^{-1})^m = -m^{-m+1} \times x^{-2}$$

$$\frac{d}{dx}x^{-m} = -m^{-m-1}$$

Which agrees with our rule

[2] The chain rule is used to differentiate an expression when there is a function within a function e.g. $(x^{-1})^m$, so if we let $u = x^{-1}$ we differentiate with respect to x to get $\frac{du}{dx} = -x^{-2}$ if we let $y = u^m$ then we can differentiate with respect to u to get $\frac{dy}{du} = mu^{m-1}$. The rule is that $\frac{dy}{du} \times \frac{du}{dx} = \frac{dy}{dx} = mu^{m-1} \times -x^{-2}$ as required

When $n = \frac{p}{q}$

Let $y = x^{\frac{p}{q}}$

Raising both sides to the power q

$$y^q = x^p$$

Differentiating this[3]

$$qy^{q-1}\frac{dy}{dx} = px^{p-1}$$

Re-arranging to give $\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{p}{q}\frac{x^{p-1}}{y^{q-1}}$$

Given that $y = x^{\frac{p}{q}}$

$$\frac{dy}{dx} = \frac{p}{q}\frac{x^{p-1}}{\left(x^{\frac{p}{q}}\right)^{q-1}}$$

Expanding the terms on the denominator

$$\frac{dy}{dx} = \frac{p}{q}\frac{x^{p-1}}{x^{p-\frac{p}{q}}}$$

This can otherwise be expressed as

$$\frac{dy}{dx} = \frac{p}{q}x^{p-1-p+\frac{p}{q}}$$

And so we have the desired result

$$\frac{dy}{dx} = \frac{p}{q}x^{\frac{p}{q}-1}$$

See also

- Product Rule
- Chain Rule
- Implicit Differentiation

References

Turner, L. K. (1976). *Advanced Mathematics – Book One*. London: Longman. pp.118-119.

[3] This is implicit differentiation, used when there is no y-term by itself on one side of the equation. You differentiate y as though it were an x-term and multiply by $\frac{dy}{dx}$ so $y^q \rightarrow qy^{q-1} \times \frac{dy}{dx}$